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## Duality for the Jordanian matrix quantum group $GL_{g,h}(2)$

B L Aneva<sup>†</sup>, V K Dobrev<sup>†‡§</sup> and S G Mihov<sup>†</sup>

<sup>†</sup> Bulgarian Academy of Sciences, Institute of Nuclear Research and Nuclear Energy, 72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria

<sup>‡</sup>International Center for Theoretical Physics, Strada Costiera 11, PO Box 586, 34100 Trieste, Italy

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**Abstract.** We find the Hopf algebra  $\mathcal{U}_{g,h}$  dual to the Jordanian matrix quantum group  $GL_{g,h}(2)$ . As an algebra it depends only on the sum of the two parameters and is split into two subalgebras:  $\mathcal{U}'_{g,h}$  (with three generators) and  $U(\mathcal{Z})$  (with one generator). The subalgebra  $U(\mathcal{Z})$  is a central Hopf subalgebra of  $\mathcal{U}_{g,h}$ . The subalgebra  $\mathcal{U}'_{g,h}$  is not a Hopf subalgebra and its co-algebra structure depends on both parameters. We discuss also two one-parameter special cases:  $g = h$  and  $g = -h$ . The subalgebra  $\mathcal{U}'_{h,h}$  is a Hopf algebra and coincides with the algebra introduced by Ohn as the dual of  $SL_h(2)$ . The subalgebra  $\mathcal{U}'_{-h,h}$  is isomorphic to  $U(sl(2))$  as an algebra but has a nontrivial co-algebra structure and again is not a Hopf subalgebra of  $\mathcal{U}_{-h,h}$ .

### 1. Introduction

The group  $GL(2)$  admits two distinct quantum group deformations with a central quantum determinant:  $GL_q(2)$  [1] and  $GL_h(2)$  [2, 3]. These are the only possible such deformations (up to isomorphism) [4]. Both may be viewed as special cases of two-parameter deformations:  $GL_{p,q}(2)$  [2] and  $GL_{g,h}(2)$  [5]. In the initial years of the development of quantum group theory mostly  $GL_q(2)$  and  $GL_{p,q}(2)$  were considered. More recently research has been started on  $SL_h(2)$  and its dual quantum algebra  $U_h(sl(2))$  [6]. In particular, aspects of differential calculus [5], and differential geometry [7] were developed, the universal  $R$ -matrix for  $U_h(sl(2))$  was given in [8–10], representations of  $U_h(sl(2))$  were constructed in [11–14], contractions of  $SL_h(2)$  and  $U_h(sl(2))$  were given in [15]. However, there are no studies until now of the two-parameter Jordanian matrix quantum group  $GL_{g,h}(2)$ . Even the dual of this algebra is not known.

This is the problem we solve in this paper. We find the Hopf algebra  $\mathcal{U}_{g,h}$  dual to the Jordanian matrix quantum group  $GL_{g,h}(2)$ . As an algebra it depends on the sum  $\tilde{g} = (g + h)/2$  of the two parameters and is split into two subalgebras:  $\mathcal{U}'_{g,h}$  (with three generators) and  $U(\mathcal{Z})$  (with one generator). The subalgebra  $U(\mathcal{Z})$  is a central Hopf subalgebra of  $\mathcal{U}_{g,h}$ . The subalgebra  $\mathcal{U}'_{g,h}$  is not a Hopf subalgebra and its coalgebra structure depends on both parameters. We discuss also two interesting one-parameter special cases:  $g = h$  and  $g = -h$ . The subalgebra  $\mathcal{U}'_{h,h}$  is a Hopf algebra and coincides with the algebra introduced by Ohn as the dual of  $SL_h(2)$ . The subalgebra  $\mathcal{U}'_{-h,h}$  is isomorphic to  $U(sl(2))$  as an algebra but has a nontrivial coalgebra structure and again is not a Hopf subalgebra of  $\mathcal{U}_{-h,h}$ .

§ E-mail address: dobrev@ictp.trieste.it

The paper is organized as follows. In section 2 we recall the group  $GL_{g,h}(2)$ . In section 3 we first recall the method (developed by one of us) of finding the dual. We then make a change of generators and introduce the appropriate PBW basis in  $GL_{g,h}(2)$ . We find the dual algebra  $\mathcal{U}_{g,h}$  and we give explicitly its algebra and coalgebra structure in propositions 1 and 2. We make a further change of basis in order to bring the algebra to a form closer to Ohn's [6]. Our main result is summarized in a theorem. We also introduce a subalgebra  $\tilde{\mathcal{U}}_{g,h}$  of  $\mathcal{U}_{g,h}$  with a basis such that no exponents of generators appear explicitly in the algebra and coalgebra relations. In section 4 we consider the two interesting one-parameter special cases:  $g = h$  and  $g = -h$  recovering the algebra of [6] in the first special case. In the appendix we apply the nonlinear map of [12] to our dual algebra.

## 2. Jordanian matrix quantum group $GL_{g,h}(2)$

In this section we recall the Jordanian two-parameter deformation  $GL_{g,h}(2)$  of  $GL(2)$  introduced in [5] (and denoted  $GL_{h,h'}$ ). One starts with a unital associative algebra generated by four elements  $a, b, c, d$  of a quantum matrix  $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with the following relations ( $g, h \in \mathbb{C}$ ):

$$\begin{aligned} [a, c] &= gc^2 & [d, c] &= hc^2 & [a, d] &= gdc - hac \\ [a, b] &= h(\mathcal{D} - a^2) & [d, b] &= g(\mathcal{D} - d^2) & [b, c] &= gdc + hac - ghc^2 \\ \mathcal{D} &= ad - bc + hac = ad - cb - gdc + ghc^2 \end{aligned} \quad (2.1)$$

where  $\mathcal{D}$  is a multiplicative quantum determinant which is not central (unless  $g = h$ ):

$$[a, \mathcal{D}] = [\mathcal{D}, d] = (g - h)\mathcal{D}c \quad [b, \mathcal{D}] = (g - h)(\mathcal{D}d - a\mathcal{D}) \quad [c, \mathcal{D}] = 0. \quad (2.2)$$

Relations (2.1) are obtained by applying either the method of Faddeev *et al* [16], namely, by solving the monodromy equation:

$$\mathbf{R}\mathbf{M}_1\mathbf{M}_2 = \mathbf{M}_2\mathbf{M}_1\mathbf{R}$$

( $\mathbf{M}_1 = \mathbf{M} \hat{\otimes} \mathbf{I}_2$ ,  $\mathbf{M}_2 = \mathbf{I}_2 \hat{\otimes} \mathbf{M}$ ), with the  $\mathbf{R}$  matrix:

$$\mathbf{R} = \begin{pmatrix} 1 & -h & h & gh \\ 0 & 1 & 0 & -g \\ 0 & 0 & 1 & g \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.3)$$

or the method of Manin [17] using  $\mathbf{M}$  as transformation matrix of the appropriate quantum planes [5].

The above algebra is turned into a bialgebra  $A_{g,h}(2)$  with the standard  $GL(2)$  co-product  $\delta$  and co-unit  $\varepsilon$ :

$$\delta \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix} \quad (2.4)$$

$$\varepsilon \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.5)$$

From (2.4), resp. (2.5) it follows:

$$\delta(\mathcal{D}) = \mathcal{D} \otimes \mathcal{D} \quad \varepsilon(\mathcal{D}) = 1. \quad (2.6)$$

Further, we shall suppose that  $\mathcal{D}$  is invertible, i.e. there is an element  $\mathcal{D}^{-1}$  which obeys:

$$\mathcal{D}\mathcal{D}^{-1} = \mathcal{D}^{-1}\mathcal{D} = 1_A \quad (\mathcal{D}^{-1}) = \mathcal{D}^{-1} \otimes \mathcal{D}^{-1} \quad \varepsilon(\mathcal{D}^{-1}) = 1. \quad (2.7)$$

(Alternatively one may say that the algebra is extended with the element  $\mathcal{D}^{-1}$ .) In this case one defines the left and right inverse matrix of  $\mathbf{M}$  [5]:

$$\begin{aligned} \mathbf{M}^{-1} &= \mathcal{D}^{-1} \begin{pmatrix} d + gc & -b + g(d - a) + g^2c \\ -c & a - gc \end{pmatrix} \\ &= \begin{pmatrix} d + hc & -b + h(d - a) + h^2c \\ -c & a - hc \end{pmatrix} \mathcal{D}^{-1}. \end{aligned} \tag{2.8}$$

The quantum group  $GL_{g,h}(2)$  is defined as the Hopf algebra obtained from the bialgebra  $A_{g,h}(2)$  when  $\mathcal{D}^{-1}$  exists and with antipode given by the formula:

$$\gamma(\mathbf{M}) = \mathbf{M}^{-1} \Rightarrow \gamma(\mathcal{D}) = \mathcal{D}^{-1} \quad \gamma(\mathcal{D}^{-1}) = \mathcal{D}. \tag{2.9}$$

For  $g = h$  one obtains from  $GL_{g,h}(2)$  the matrix quantum group  $GL_h(2) = GL_{h,h}(2)$ , and, if the condition  $\mathcal{D} = 1_{\mathcal{A}}$  holds, the matrix quantum group  $SL_h(2)$ . Analogously, for  $g = h = 0$  one obtains from  $GL_{g,h}(2)$  the algebra of functions over the classical groups  $GL(2)$  and  $SL(2)$ , resp.

### 3. The dual of $GL_{g,h}(2)$

#### 3.1. Summary of the method

Two bialgebras  $\mathcal{U}, \mathcal{A}$  are said to be *in duality* [18] if there exists a doubly nondegenerate bilinear form

$$\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{A} \rightarrow \mathbb{C} \quad \langle \cdot, \cdot \rangle : (u, a) \mapsto \langle u, a \rangle \quad u \in \mathcal{U} \quad a \in \mathcal{A} \tag{3.1}$$

such that, for  $u, v \in \mathcal{U}, a, b \in \mathcal{A}$ :

$$\langle u, ab \rangle = \langle \delta_{\mathcal{U}}(u), a \otimes b \rangle \quad \langle uv, a \rangle = \langle u \otimes v, \delta_{\mathcal{A}}(a) \rangle \tag{3.2a}$$

$$\langle 1_{\mathcal{U}}, a \rangle = \varepsilon_{\mathcal{A}}(a) \quad \langle u, 1_{\mathcal{A}} \rangle = \varepsilon_{\mathcal{U}}(u). \tag{3.2b}$$

Two Hopf algebras  $\mathcal{U}, \mathcal{A}$  are said to be *in duality* [18] if they are in duality as bialgebras and if

$$\langle \gamma_{\mathcal{U}}(u), a \rangle = \langle u, \gamma_{\mathcal{A}}(a) \rangle. \tag{3.2c}$$

It is enough to define the pairing (3.1) between the generating elements of the two algebras. The pairing between any other elements of  $\mathcal{U}, \mathcal{A}$  follows then from relations (3.2) and the standard bilinear form inherited by the tensor product.

The duality between two bialgebras or Hopf algebras may be used also to obtain the unknown dual of a known algebra; for that it is enough to give the pairing between the generating elements of the unknown algebra with arbitrary elements of the PBW basis of the known algebra. Using these initial pairings and the duality properties one may find the unknown algebra. Such an approach was first given by Sudbery [19]. He obtained  $U_q(sl(2)) \otimes U(u(1))$  as the algebra of tangent vectors at the identity of  $GL_q(2)$ . The initial pairings were defined through the tangent vectors at the identity. However, such calculations become very difficult for more complicated algebras. Thus, in [20] a generalization was proposed in which the initial pairings are postulated to be equal to the classical undeformed results. This generalized method was applied in [20] to the standard two-parameter deformation  $GL_{p,q}(2)$ , (where also Sudbery's method was used), then in [21] to the multiparameter deformation of  $GL(n)$ , and in [22] to the matrix quantum Lorentz group of [23]. One should note that the dual of  $GL_{p,q}(2)$  was obtained also in [24] by methods of  $q$ -differential calculus.

3.2. Change of basis for  $GL_{g,h}(2)$  and generators of the dual algebra

In the present paper we apply the method of [20] to find the dual of  $GL_{g,h}(2)$ . Following [20] we first need to fix a PBW basis of  $GL_{g,h}(2)$ . At first one may be inclined to use a PBW basis as the one introduced in [6] for the case  $g = h$ , namely consisting of all monomials  $a^k d^\ell b^m c^n$ , where  $k, \ell, m, n \in \mathbb{Z}_+$ . (Actually, the basis in [6] is for  $SL_h(2)$  and is obtained by restricting to indices fulfilling  $k\ell = 0$ .) However, the calculation with such a basis are more difficult. Our analysis showed that it would be simpler to work with the following PBW basis:

$$a^k d^\ell c^n b^m \quad k, \ell, m, n \in \mathbb{Z}_+ \tag{3.3}$$

the explanation being that the elements  $a, d, c$  generate a subalgebra (though not a Hopf subalgebra) of  $GL_{g,h}(2)$ , cf the first line of (2.1). Further simplification results if we make the following change of generating elements and parameters:

$$\begin{aligned} \tilde{a} &= \frac{1}{2}(a + d) & \tilde{d} &= \frac{1}{2}(a - d) \\ \tilde{g} &= \frac{1}{2}(g + h) & \tilde{h} &= \frac{1}{2}(g - h). \end{aligned} \tag{3.4}$$

With these generating elements and parameters the algebra relations become:

$$\begin{aligned} c\tilde{a} &= \tilde{a}c - \tilde{g}c^2 & c\tilde{d} &= \tilde{d}c - \tilde{h}c^2 & \tilde{d}\tilde{a} &= \tilde{a}\tilde{d} - \tilde{g}\tilde{d}c + \tilde{h}\tilde{a}c \\ b\tilde{a} &= \tilde{a}b + \tilde{g}cb - 2\tilde{h}\tilde{a}\tilde{d} + 2\tilde{g}\tilde{d}^2 + (\tilde{g}^2 - \tilde{h}^2)\tilde{a}c + \tilde{g}(\tilde{h}^2 - \tilde{g}^2)c^2 \\ b\tilde{d} &= \tilde{d}b - \tilde{h}cb + 2\tilde{g}\tilde{a}\tilde{d} - 2\tilde{h}\tilde{d}^2 + (\tilde{h}^2 - \tilde{g}^2)\tilde{d}c + \tilde{h}(\tilde{g}^2 - \tilde{h}^2)c^2 \\ bc &= cb + 2\tilde{g}\tilde{a}c - 2\tilde{h}\tilde{d}c + (\tilde{h}^2 - \tilde{g}^2)c^2 \\ \mathcal{D} &= \tilde{a}^2 - \tilde{d}^2 - cb + (\tilde{g}^2 - \tilde{h}^2)c^2 - \tilde{g}\tilde{a}c + \tilde{h}\tilde{d}c. \end{aligned} \tag{3.5}$$

Note that these relations are written in anticipation of the PBW basis:

$$f = f_{k,\ell,m,n} = \tilde{a}^k \tilde{d}^\ell c^n b^m \quad k, \ell, m, n \in \mathbb{Z}_+. \tag{3.6}$$

Note also that the relations in the subalgebras generated by  $a, d, c$  and  $\tilde{a}, \tilde{d}, c$  are isomorphic under the change:  $a \mapsto \tilde{a}, d \mapsto \tilde{d}, c \mapsto c, g \mapsto \tilde{g}, h \mapsto \tilde{h}$ , cf the first lines in (2.1) and (3.5).

The co-algebra relations become:

$$\delta \left( \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} \right) = \begin{pmatrix} \tilde{a} \otimes \tilde{a} + \tilde{d} \otimes \tilde{d} + \frac{1}{2}b \otimes c + \frac{1}{2}c \otimes b & \tilde{a} \otimes b + \tilde{d} \otimes b + b \otimes \tilde{a} - b \otimes \tilde{d} \\ c \otimes \tilde{a} + c \otimes \tilde{d} + \tilde{a} \otimes c - \tilde{d} \otimes c & \tilde{a} \otimes \tilde{d} + \tilde{d} \otimes \tilde{a} + \frac{1}{2}b \otimes c - \frac{1}{2}c \otimes b \end{pmatrix} \tag{3.7}$$

$$\varepsilon \left( \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.8}$$

$$\begin{aligned} \gamma \left( \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} \right) &= \mathcal{D}^{-1} \begin{pmatrix} \tilde{a} - \tilde{d} + (\tilde{g} + \tilde{h})c & -b - 2(\tilde{g} + \tilde{h})\tilde{d} + (\tilde{g} + \tilde{h})^2c \\ -c & \tilde{a} + \tilde{d} - (\tilde{g} + \tilde{h})c \end{pmatrix} \\ &= \begin{pmatrix} \tilde{a} - \tilde{d} + (\tilde{g} - \tilde{h})c & -b + 2(\tilde{h} - \tilde{g})\tilde{d} + (\tilde{g} - \tilde{h})^2c \\ -c & \tilde{a} + \tilde{d} + (\tilde{h} - \tilde{g})c \end{pmatrix} \mathcal{D}^{-1}. \end{aligned} \tag{3.9}$$

Let us denote by  $\mathcal{U}_{g,h} = U_{g,h}(gl(2))$  the unknown yet dual algebra of  $GL_{g,h}(2)$ , and by  $A, B, C, D$  the four generators of  $\mathcal{U}_{g,h}$ . Following [20] we shall define the pairing  $\langle Z, f \rangle$ ,  $Z = A, B, C, D$ ,  $f$  is from (3.6), as the classical tangent vector at the identity:

$$\langle Z, f \rangle \equiv \varepsilon \left( \frac{\partial f}{\partial y} \right) \quad (Z, y) = (A, \tilde{a}), (B, b), (C, c), (D, \tilde{d}). \tag{3.10}$$

From this we obtain the explicit expressions:

$$\langle A, f \rangle = \varepsilon \left( \frac{\partial f}{\partial \tilde{a}} \right) = k\delta_{\ell 0}\delta_{m 0}\delta_{n 0} \tag{3.11a}$$

$$\langle B, f \rangle = \varepsilon \left( \frac{\partial f}{\partial b} \right) = \delta_{\ell 0}\delta_{m 1}\delta_{n 0} \tag{3.11b}$$

$$\langle C, f \rangle = \varepsilon \left( \frac{\partial f}{\partial c} \right) = \delta_{\ell 0}\delta_{m 0}\delta_{n 1} \tag{3.11c}$$

$$\langle D, f \rangle = \varepsilon \left( \frac{\partial f}{\partial \tilde{d}} \right) = \delta_{\ell 1}\delta_{m 0}\delta_{n 0}. \tag{3.11d}$$

### 3.3. Algebra structure of the dual

First we find the commutation relations between the generators of  $\mathcal{U}_{g,h}$ . Below we shall need expressions like  $e^{\nu B}$  which we define as formal power series  $e^{\nu B} = 1_{\mathcal{U}} + \sum_{p \in \mathbb{N}} \frac{\nu^p}{p!} B^p$ . We have:

*Proposition 1.* The commutation relations of the generators  $A, B, C, D$  introduced by (3.11) are:

$$[B, C] = D \tag{3.12a}$$

$$[D, B] = \frac{1}{\tilde{g}}(e^{2\tilde{g}B} - 1_{\mathcal{U}}) \tag{3.12b}$$

$$[D, C] = -2C + \tilde{g}D^2 - \tilde{g}A \tag{3.12c}$$

$$[A, B] = 0 \quad [A, C] = 0 \quad [A, D] = 0. \tag{3.12d}$$

*Proof.* Using the assumed duality the above relations are shown by calculating their pairings with the basis monomials  $f = \tilde{a}^k \tilde{d}^\ell c^n b^m$  of the dual algebra. In particular, the pairing of  $f = \tilde{a}^k \tilde{d}^\ell c^n b^m$  with the commutators is:

$$\langle [B, C], f \rangle = \delta_{\ell 1}\delta_{m 0}\delta_{n 0} \tag{3.13a}$$

$$\langle [D, B], f \rangle = \delta_{\ell 0}\theta_{m 1}\delta_{n 0}2^m \tilde{g}^{m-1} \tag{3.13b}$$

$$\langle [D, C], f \rangle = -2\delta_{\ell 0}\delta_{m 0}\delta_{n 1} + 2\tilde{g}\delta_{\ell 2}\delta_{m 0}\delta_{n 0} \tag{3.13c}$$

$$\langle [A, B], f \rangle = \langle [A, C], f \rangle = \langle [A, D], f \rangle = 0 \tag{3.13d}$$

$$\theta_{rs} \equiv \begin{cases} 1 & r \geq s \\ 0 & r < s. \end{cases} \tag{3.13e}$$

To calculate a commutator  $\langle [W, Z], f \rangle$  one first calculates  $\langle WZ, f \rangle$  and  $\langle ZW, f \rangle$ . The pairing of any quadratic monomial of the unknown dual algebra with  $f = \tilde{a}^k \tilde{d}^\ell c^n b^m$  is given by the duality properties (3.2):

$$\langle WZ, f \rangle = \langle W \otimes Z, \delta_A(f) \rangle = \left\langle W \otimes Z, \sum_j f'_j \otimes f''_j \right\rangle = \sum_j \langle W, f'_j \rangle \langle Z, f''_j \rangle \tag{3.14}$$

where  $f'_j, f''_j$  are elements of the basis (3.6) and so further a direct application of (3.11) is used. We should note that these calculations though complicated do not require explicit knowledge of  $\delta_A(f) = \sum_j f'_j \otimes f''_j$  for all  $f$ , and furthermore not all terms in the sums are necessary. In particular, while calculating  $\delta_A(f)$  one may neglect terms containing the element  $c$  on either side of the tensor sign in second and higher degrees even before reordering the terms to the basis monomials, since from the commutation relations it is clear

that those terms will not produce any term with  $c$  in zero or first degree, and any  $f'_j$  ( $f''_j$ ) containing  $c$  in second and higher degrees will give zero in (3.14) for any  $W$  ( $Z$ ). For the same reasons, if  $W \neq C$  ( $Z \neq C$ ) one may neglect terms containing the element  $c$  on the left (right) side of the tensor sign in the first degree even before reordering the terms to the basis monomials. Similar reasons hold for the elements  $\tilde{d}$ . Taking into account such simplifications one may find the pairings of the quadratic monomials necessary for (3.13), e.g.

$$\langle BC, f \rangle = \frac{1}{2} \delta_{\ell 1} \delta_{m 0} \delta_{n 0} + \tilde{h} \delta_{\ell 1} \delta_{m 1} \delta_{n 0} + \delta_{\ell 0} \delta_{n 0} \theta_{m 2} \frac{1}{2} (\tilde{g}^2 - \tilde{h}^2) \tilde{g}^{m-2} + \delta_{\ell 0} \delta_{m 1} \delta_{n 1} \quad (3.15a)$$

$$\langle CB, f \rangle = -\frac{1}{2} \delta_{\ell 1} \delta_{m 0} \delta_{n 0} + \tilde{h} \delta_{\ell 1} \delta_{m 1} \delta_{n 0} + \delta_{\ell 0} \delta_{n 0} \theta_{m 2} \frac{1}{2} (\tilde{g}^2 - \tilde{h}^2) \tilde{g}^{m-2} + \delta_{\ell 0} \delta_{m 1} \delta_{n 1} \quad (3.15b)$$

$$\langle DB, f \rangle = \delta_{\ell 0} \delta_{n 0} (\delta_{m 1} + \theta_{m 2} 2^{m-1} \tilde{g}^{m-2} (\tilde{g} - \tilde{h})) \quad (3.15c)$$

$$\langle BD, f \rangle = -\delta_{\ell 0} \delta_{n 0} (\delta_{m 1} + \theta_{m 2} 2^{m-1} \tilde{g}^{m-2} (\tilde{g} + \tilde{h})) \quad (3.15d)$$

$$\langle DC, f \rangle = -\delta_{\ell 0} \delta_{m 0} \delta_{n 1} + (\tilde{h} + \tilde{g}) \delta_{\ell 2} \delta_{m 0} \delta_{n 0} + k \tilde{g} \delta_{\ell 1} \delta_{m 0} \delta_{n 0} + \delta_{\ell 1} \delta_{m 0} \delta_{n 1} \quad (3.15e)$$

$$\langle CD, f \rangle = \delta_{\ell 0} \delta_{m 0} \delta_{n 1} + (\tilde{h} - \tilde{g}) \delta_{\ell 2} \delta_{m 0} \delta_{n 0} + k \tilde{g} \delta_{\ell 1} \delta_{m 0} \delta_{n 0} + \delta_{\ell 1} \delta_{m 0} \delta_{n 1}. \quad (3.15f)$$

Note that quadratic relations (3.15) depend on both parameters, while the commutation relations (3.13), which follow from (3.15), depend only on the parameter  $\tilde{g}$ .

Now in order to establish (3.12a) it is enough to compare the RHS of (3.13a) and (3.11d). Further, for relation (3.12b) we use (3.13b) and:

$$\langle B^p, f \rangle = p! \delta_{\ell 0} \delta_{m p} \delta_{n 0} \quad (3.15h)$$

(proved by induction) and its consequence:

$$\langle (e^{2\tilde{g}B} - 1_{\mathcal{U}}), f \rangle = \sum_{p \in \mathbb{N}} \frac{(2\tilde{g})^p}{p!} \langle B^p, f \rangle = \sum_{p \in \mathbb{N}} \frac{(2\tilde{g})^p}{p!} p! \delta_{\ell 0} \delta_{m p} \delta_{n 0} = (2\tilde{g})^m \delta_{\ell 0} \theta_{m 1} \delta_{n 0}. \quad (3.15i)$$

To establish (3.12c) we compare the RHS of (3.13c) with the appropriate linear combination of the RHS of three equations, namely (3.11a), (3.11c) and

$$\langle D^2, f \rangle = 2\delta_{\ell 2} \delta_{m 0} \delta_{n 0} + k\delta_{\ell 0} \delta_{m 0} \delta_{n 0}. \quad (3.15g)$$

This finishes the proof.  $\square$

Note that the commutation relations (3.12) depend only on the parameter  $\tilde{g}$  and that the generator  $A$  is central. This is similar to the situation of the dual algebra  $\mathcal{U}_{p,q}$  of the standard matrix quantum group  $GL_{p,q}$  the commutation relations of which depend only on the combination  $q' = \sqrt{pq}$  and also one generator is central [24, 20]. Here the central generator appears as a central extension but this is fictitious since this may be corrected by a change of basis, namely, by replacing the generator  $C$  by a generator  $\tilde{C}$ :

$$C = \tilde{C} - \frac{\tilde{g}}{2} A. \quad (3.16)$$

With this only (3.12c) changes to:

$$[D, \tilde{C}] = -2\tilde{C} + \tilde{g}D^2. \quad (3.12c')$$

Besides this change we shall make a change of generating elements of  $\mathcal{U}_{g,h}$  in order to bring the commutation relations to a form closer to the algebra of [6]. Thus, we make the following substitutions:

$$D = e^{\mu B} H e^{v B} \quad (3.17a)$$

$$C = e^{\mu' B} Y e^{v' B} - \frac{\tilde{g}}{2} \sinh(\tilde{g}B) e^{(\mu'+v')B} - \frac{\tilde{g}}{2} A. \quad (3.17b)$$

Substituting (3.17) into (3.12a) we obtain the desired result  $[B, Y] = H$  if we choose:  $\mu' = \mu$ ,  $\nu' = \nu$ . Substituting (3.17) into (3.12b) we obtain the desired result  $[H, B] = \frac{2}{\tilde{g}} \sinh(\tilde{g}B)$  if we choose:  $\mu + \nu = \tilde{g}$ . Thus with conditions:

$$\mu + \nu = \tilde{g} \quad \mu' = \mu \quad \nu' = \nu \tag{3.17c}$$

we obtain the following commutation relations instead of (3.12):

$$[B, Y] = H \tag{3.18a}$$

$$[H, B] = \frac{2}{\tilde{g}} \sinh(\tilde{g}B) \tag{3.18b}$$

$$[H, Y] = -Y \cosh(\tilde{g}B) - \cosh(\tilde{g}B)Y \\ = -2Y \cosh(\tilde{g}B) - \tilde{g}H \sinh(\tilde{g}B) + \tilde{g} \sinh(\tilde{g}B) \cosh(\tilde{g}B) \tag{3.18c}$$

$$[A, B] = 0 \quad [A, Y] = 0 \quad [A, H] = 0. \tag{3.18d}$$

Note that relations (3.18a), (3.18b), (3.18c) coincide with those of the one-parameter algebra of [6], (cf section 4.1), though the coalgebra structure is different as we shall see below. We can use this coincidence to derive the Casimir operator of  $\mathcal{U}_{g,h}$ :

$$\hat{\mathcal{C}}_2 = f_1(A)\mathcal{C}_2 + f_2(A) \\ \mathcal{C}_2 = \frac{1}{2}(H^2 + \sinh^2(\tilde{g}B)) + \frac{1}{\tilde{g}}(Y \sinh(\tilde{g}B) + \sinh(\tilde{g}B)Y) \tag{3.19}$$

where  $f_1(A)$ ,  $f_2(A)$  are arbitrary polynomials in the central generator  $A$ . To derive (3.19) it is enough to check that  $[\mathcal{C}_2, Z] = 0$  for  $Z = B, Y, H$ . The latter follows also from the fact [25] that  $\mathcal{C}_2$  is the Casimir of the one-parameter algebra of [6].

Finally we also write a subalgebra  $\tilde{\mathcal{U}}_{g,h}$  of  $\mathcal{U}_{g,h}$  with the basis:  $A, K = e^{\tilde{g}B} = K^+, K^{-1} = e^{-\tilde{g}B} = K^-, Y, H$ , so that in terms of  $A, K, K^{-1}, Y, H$  no exponents of generators appear in the algebra and coalgebra relations. Thus instead of (3.18) we have:

$$[K^\pm, Y] = \pm \tilde{g} H K^\pm \pm \frac{\tilde{g}}{2}(1_{\mathcal{U}} - K^{\pm 2}) \tag{3.20a}$$

$$[H, K^\pm] = K^{\pm 2} - 1_{\mathcal{U}} \tag{3.20b}$$

$$[H, Y] = -Y(K + K^{-1}) + \frac{\tilde{g}}{2}H(K^{-1} - K) + \frac{\tilde{g}}{4}(K^2 - K^{-2}) \tag{3.20c}$$

$$K K^{-1} = K^{-1} K = 1_{\mathcal{U}} \tag{3.20c'}$$

$$[A, K] = [A, K^{-1}] = 0 \quad [A, Y] = 0 \quad [A, H] = 0. \tag{3.20d}$$

### 3.4. Co-algebra structure of the dual

We turn now to the coalgebra structure of  $\mathcal{U}_{g,h}$ . We have:

*Proposition 2.* (i) The comultiplication in the algebra  $\mathcal{U}_{g,h}$  is given by:

$$\delta_{\mathcal{U}}(A) = A \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes A \tag{3.21a}$$

$$\delta_{\mathcal{U}}(B) = B \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes B \tag{3.21b}$$

$$\delta_{\mathcal{U}}(Y) = Y \otimes e^{-\tilde{g}B} + e^{\tilde{g}B} \otimes Y - \frac{\tilde{h}^2}{\tilde{g}} \sinh(\tilde{g}B) \otimes A^2 e^{-\tilde{g}B} + \tilde{h}H \otimes A e^{-\tilde{g}B} \tag{3.21c}$$

$$\delta_{\mathcal{U}}(H) = H \otimes e^{-\tilde{g}B} + e^{\tilde{g}B} \otimes H - \frac{2\tilde{h}}{\tilde{g}} \sinh(\tilde{g}B) \otimes A e^{-\tilde{g}B}. \tag{3.21d}$$

(ii) The co-unit relations in  $\mathcal{U}_{g,h}$  are given by:

$$\varepsilon_{\mathcal{U}}(Z) = 0 \quad Z = A, B, Y, H. \quad (3.22)$$

(iii) The antipode in the algebra  $\mathcal{U}_{g,h}$  is given by:

$$\gamma_{\mathcal{U}}(A) = -A \quad (3.23a)$$

$$\gamma_{\mathcal{U}}(B) = -B \quad (3.23b)$$

$$\gamma_{\mathcal{U}}(Y) = -e^{-\tilde{g}B} Y e^{\tilde{g}B} + \frac{\tilde{h}^2}{\tilde{g}} \sinh(\tilde{g}B) A^2 + \tilde{h} e^{-\tilde{g}B} H A e^{\tilde{g}B} \quad (3.23c)$$

$$\gamma_{\mathcal{U}}(H) = -e^{-\tilde{g}B} H e^{\tilde{g}B} - \frac{2\tilde{h}}{\tilde{g}} \sinh(\tilde{g}B) A. \quad (3.23d)$$

*Proof.* (i) We use the duality property (3.2a), namely we have

$$\langle Z, f_1 f_2 \rangle = \langle \delta_{\mathcal{U}}(Z), f_1 \otimes f_2 \rangle$$

for every generator  $Z$  of  $\mathcal{U}_{g,h}$  and for every  $f_1, f_2 \in GL_{g,h}(2)$ . Then we calculate separately the LHS and RHS and comparing the results prove (3.21). The check of (3.21a), (3.21b) is easy. Instead of (3.21c), (3.21d) we first find the coproduct of the original generators  $C, D$ :

$$\delta_{\mathcal{U}}(C) = C \otimes 1_{\mathcal{U}} + e^{2\tilde{g}B} \otimes C + \frac{1}{2\tilde{g}} (e^{2\tilde{g}B} - 1_{\mathcal{U}}) \otimes (\tilde{g}^2 A - \tilde{h}^2 A^2) + \tilde{h} D \otimes A \quad (3.21c')$$

$$\delta_{\mathcal{U}}(D) = D \otimes 1_{\mathcal{U}} + e^{2\tilde{g}B} \otimes D + \frac{\tilde{h}}{\tilde{g}} (1_{\mathcal{U}} - e^{2\tilde{g}B}) \otimes A \quad (3.21d')$$

and then (3.21c), (3.21d) follow. To check (3.21c') the following choices are crucial:  $(f_1, f_2) = (\tilde{a}^k \tilde{d}^l c^n b^m, 1_{\mathcal{U}})$ ,  $(b^m, c)$ ,  $(b^m, a^k)$ ,  $(d^l, a^k)$ . To check (3.21d') the crucial choices are:  $(f_1, f_2) = (\tilde{a}^k \tilde{d}^l c^n b^m, 1_{\mathcal{U}})$ ,  $(b^m, d)$ ,  $(b^m, a^k)$ .

(ii) Formulae (3.22) follow from  $\varepsilon_{\mathcal{U}}(Z) = \langle Z, 1_A \rangle$ , cf (3.2b), and using the defining relations (3.11).

(iii) Formulae (3.23) follow from (3.2c) or by using the following Hopf algebra axiom [18]:

$$m \circ (\text{id} \otimes \gamma_{\mathcal{U}}) \circ \delta_{\mathcal{U}} = i \circ \varepsilon_{\mathcal{U}} \quad (3.24)$$

where  $m$  is the usual product in the algebra:  $m(Z \otimes W) = ZW$ ,  $Z, W \in \mathcal{U}$  and  $i$  is the natural embedding of  $F$  into  $\mathcal{U}$ :  $i(v) = v 1_{\mathcal{U}}$ ,  $v \in F$ . This is applied in our case with  $\mathcal{U} \mapsto \mathcal{U}_{g,h}$ ,  $F = \mathbb{C}$ , to the elements  $A, B, Y, H$  and using (3.21) and (3.22).  $\square$

*Corollary 1.* For later reference we mention also the coproduct and antipode of the intermediate generator  $\tilde{C}$  and the antipode of the initial generator  $D$ :

$$\delta_{\mathcal{U}}(\tilde{C}) = \tilde{C} \otimes 1_{\mathcal{U}} + e^{2\tilde{g}B} \otimes \tilde{C} - \frac{\tilde{h}^2}{2\tilde{g}} (e^{2\tilde{g}B} - 1_{\mathcal{U}}) \otimes A^2 + \tilde{h} D \otimes A \quad (3.21c'')$$

$$\gamma_{\mathcal{U}}(\tilde{C}) = -e^{-2\tilde{g}B} \tilde{C} + \frac{\tilde{h}^2}{2\tilde{g}} (1_{\mathcal{U}} - e^{-2\tilde{g}B}) A^2 + \tilde{h} e^{-2\tilde{g}B} D A \quad (3.23c')$$

$$\gamma_{\mathcal{U}}(D) = -e^{-2\tilde{g}B} D + \frac{\tilde{h}}{\tilde{g}} (e^{-2\tilde{g}B} - 1_{\mathcal{U}}) A. \quad (3.23d')$$

Corollary 2. The coalgebra structure in the subalgebra  $\tilde{\mathcal{U}}_{g,h}$  is given as follows:

(i) comultiplication:

$$\delta_{\mathcal{U}}(A) = A \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes A \tag{3.25a}$$

$$\delta_{\mathcal{U}}(K^{\pm}) = K^{\pm} \otimes K^{\pm} \tag{3.25b}$$

$$\delta_{\mathcal{U}}(Y) = Y \otimes K^{-1} + K \otimes Y - \frac{\tilde{h}^2}{2\tilde{g}}(K - K^{-1}) \otimes A^2 K^{-1} + \tilde{h}H \otimes AK^{-1} \tag{3.25c}$$

$$\delta_{\mathcal{U}}(H) = H \otimes K^{-1} + K \otimes H + \frac{\tilde{h}}{\tilde{g}}(K^{-1} - K) \otimes AK^{-1} \tag{3.25d}$$

(ii) co-unit:

$$\varepsilon_{\mathcal{U}}(Z) = 0 \quad Z = A, Y, H \quad \varepsilon_{\mathcal{U}}(Z) = 1 \quad Z = K, K^{-1} \tag{3.26}$$

(iii) antipode:

$$\gamma_{\mathcal{U}}(A) = -A \tag{3.27a}$$

$$\gamma_{\mathcal{U}}(K^{\pm}) = K^{\mp} \tag{3.27b}$$

$$\gamma_{\mathcal{U}}(Y) = -K^{-1}YK + \frac{\tilde{h}^2}{2\tilde{g}}(K - K^{-1})A^2 + \tilde{h}K^{-1}HAK \tag{3.27c}$$

$$\gamma_{\mathcal{U}}(H) = -K^{-1}HK + \frac{\tilde{h}}{\tilde{g}}(K^{-1} - K)A. \tag{3.27d}$$

### 3.5. Main result

Finally we can state the following:

*Theorem 1.* The Hopf algebra  $\mathcal{U}_{g,h}$  dual to  $GL_{g,h}(2)$  is generated by  $A, B, Y, H$  (or  $A, K, K^{-1}, Y, H$ ), cf relations (3.11) and (3.17). It is given by relations (3.18), (3.21), (3.22), (3.23), (resp. (3.20), (3.25), (3.26), (3.27)). As an algebra it depends only on one parameter  $\tilde{g} = (g + h)/2$  and is split into two subalgebras:  $\mathcal{U}'_{g,h}$  (resp.  $\tilde{\mathcal{U}}'_{g,h}$ ) generated by  $B, Y, H$  (resp.  $K, K^{-1}, Y, H$ ) and  $U(\mathcal{Z})$ , where the algebra  $\mathcal{Z}$  is spanned by  $A$ . The subalgebra  $U(\mathcal{Z})$  is central in  $\mathcal{U}_{g,h}$  and is also a Hopf subalgebra of  $\mathcal{U}_{g,h}$ . The subalgebra  $\mathcal{U}'_{g,h}$  (resp.  $\tilde{\mathcal{U}}'_{g,h}$ ) is not a Hopf subalgebra.

*Proof.* Actually this statement is summarizing our results in this section, cf propositions 1 and 2, and the basis change (3.17). It remains only to note that  $U(\mathcal{Z})$  is a Hopf subalgebra since  $A$  commutes with the other generators and its Hopf algebra operations are in terms of  $A$  itself. The subalgebra generated by  $\mathcal{U}'_{g,h}$  (resp.  $\tilde{\mathcal{U}}'_{g,h}$ ) is not a Hopf subalgebra since the generator  $A$  takes part in formulae (3.21c), (3.21d), (3.23c), (3.23d) (resp. (3.25c), (3.25d), (3.27c), (3.27d)).  $\square$

## 4. One-parameter cases

It is interesting to discuss the one-parameter special cases of the matrix quantum group  $GL_{g,h}(2)$  and its dual.

4.1. Case  $g = h$ 

The one-parameter matrix quantum group  $GL_{\tilde{g}}(2)$  [2, 3], is obtained from  $GL_{g,h}(2)$  by setting  $g = h = \tilde{g}$ . Thus the dual algebra  $\mathcal{U}_{\tilde{g}} \equiv \mathcal{U}_{\tilde{g},\tilde{g}}$  of  $GL_{\tilde{g}}(2)$  is obtained by setting  $\tilde{h} = \frac{1}{2}(g - h) = 0$  in (3.18), (3.21), (3.22), (3.23). Since the commutation relations (3.18) and the co-unit relations (3.22) do not depend on  $\tilde{h}$  they remain unchanged for  $\mathcal{U}_{\tilde{g}}$ . The coproduct and antipode relations of  $\mathcal{U}_{\tilde{g}}$  are:

$$\delta_{\mathcal{U}}(A) = A \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes A \quad (4.1a)$$

$$\delta_{\mathcal{U}}(B) = B \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes B \quad (4.1b)$$

$$\delta_{\mathcal{U}}(Y) = Y \otimes e^{-\tilde{g}B} + e^{\tilde{g}B} \otimes Y \quad (4.1c)$$

$$\delta_{\mathcal{U}}(H) = H \otimes e^{-\tilde{g}B} + e^{\tilde{g}B} \otimes H \quad (4.1d)$$

$$\gamma_{\mathcal{U}}(A) = -A \quad (4.2a)$$

$$\gamma_{\mathcal{U}}(B) = -B \quad (4.2b)$$

$$\gamma_{\mathcal{U}}(Y) = -e^{-\tilde{g}B} Y e^{\tilde{g}B} \quad (4.2c)$$

$$\gamma_{\mathcal{U}}(H) = -e^{-\tilde{g}B} H e^{\tilde{g}B}. \quad (4.2d)$$

We see that the one-parameter Hopf algebra  $\mathcal{U}_{\tilde{g}}$  is split into two Hopf subalgebras  $\mathcal{U}'_{\tilde{g}} \equiv \mathcal{U}'_{\tilde{g},\tilde{g}}$  and  $U(\mathcal{Z})$  and we may write:

$$\mathcal{U}_{\tilde{g}} = \mathcal{U}'_{\tilde{g}} \otimes U(\mathcal{Z}). \quad (4.3)$$

Now we compare the algebra  $\mathcal{U}'_{\tilde{g}}$  with the algebra of [6]. We see that after the identification  $B \mapsto X$ ,  $\tilde{g} \mapsto -h$ , the algebra relations (3.18a), (3.18b), (3.18c) and the coalgebra relations (4.1b), (4.1c), (4.1d), (3.22), (4.2b), (4.2c), (4.2d) coincide with their counterparts in [6], i.e. the algebra  $\mathcal{U}'_{\tilde{g}}$  coincides with the algebra of Ohn. We also note that the algebra  $\mathcal{U}'_{\tilde{g}}$  in the basis  $B, \tilde{C}, D$  (cf (3.12a), (3.12b), (3.12c'), (3.21b), (3.21c''), (3.21d'), (3.22), (3.23b), (3.23c'), (3.23d')) coincides for  $\tilde{h} = 0$  with the version given in [10] after the identification:  $(B, \tilde{C}, D; \tilde{g}) \mapsto (A_+, A_-, A; z)$ , and by using the opposite coalgebra structure.

4.2. Case  $g = -h$ 

Here we consider another one-parameter case:  $g = -h = \tilde{h}$ , i.e.  $\tilde{g} = 0$ . From (3.18), (3.21), (3.23), we obtain:

$$[B, Y] = H \quad (4.4a)$$

$$[H, B] = 2B \quad (4.4b)$$

$$[H, Y] = -2Y \quad (4.4c)$$

$$[A, B] = 0 \quad [A, Y] = 0 \quad [A, H] = 0 \quad (4.4d)$$

$$\delta_{\mathcal{U}}(A) = A \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes A \quad (4.5a)$$

$$\delta_{\mathcal{U}}(B) = B \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes B \quad (4.5b)$$

$$\delta_{\mathcal{U}}(Y) = Y \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes Y - \tilde{h}^2 B \otimes A^2 + \tilde{h} H \otimes A \quad (4.5c)$$

$$\delta_{\mathcal{U}}(H) = H \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes H - 2\tilde{h} B \otimes A \quad (4.5d)$$

$$\gamma_{\mathcal{U}}(A) = -A \quad (4.6a)$$

$$\gamma_{\mathcal{U}}(B) = -B \quad (4.6b)$$

$$\gamma_{\mathcal{U}}(Y) = -Y + \tilde{h}^2 B A^2 + \tilde{h} H A \quad (4.6c)$$

$$\gamma_{\mathcal{U}}(H) = -H - 2\tilde{h} B A. \quad (4.6d)$$

Thus, for  $\tilde{g} = 0$  the interesting feature is that the subalgebra  $\mathcal{U}'_{\tilde{h}, -\tilde{h}}$  is isomorphic to the undeformed  $U(sl(2))$  with  $sl(2)$  spanned by  $B, Y, H$ . However, as in the general case, the coalgebra sector is not classical, and the generators  $B, Y, H$  do not close a co-subalgebra.

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**Appendix. Application of a nonlinear map**

In [12] a nonlinear map was proposed under which the one-parameter Ohn’s algebra was brought to undeformed  $sl(2)$  form, though, the coalgebra structure becomes even more complicated, cf [13] and [14]. Since our two-parameter dual is like Ohn’s algebra in the algebra sector we can also apply the map of [12], which we do in this appendix. We give the map in our notation, namely, following (28) and (33) of [12] we set:

$$I_+ = \frac{2}{\tilde{g}} \tanh\left(\frac{\tilde{g}B}{2}\right) = -\frac{2}{\tilde{g}}\left(1_{\mathcal{U}} + 2\sum_{\ell=1}^{\infty}(-K)^{\ell}\right)\left(= \frac{2}{\tilde{g}}\left(\frac{K-1_{\mathcal{U}}}{K+1_{\mathcal{U}}}\right)\right) \tag{A.1a}$$

$$I_- = \cosh\left(\frac{\tilde{g}B}{2}\right)Y \cosh\left(\frac{\tilde{g}B}{2}\right) = \frac{1}{4}(K^{1/2} + K^{-1/2})Y (K^{1/2} + K^{-1/2}). \tag{A.1b}$$

Then we have, as in [12] for the case  $U_h(sl(2))$ , (note though that we do not rescale  $H$ ) the classical  $gl(2)$  commutation relations and Casimir:

$$[H, I_{\pm}] = \pm 2I_{\pm} \quad [I_+, I_-] = H \quad [A, I_{\pm}] = [A, H] = 0 \tag{A.2}$$

$$\hat{C}_2^c = f_1(A)C_2^c + f_2(A) \quad C_2^c = I_+I_- + I_-I_+ + \frac{1}{2}H^2. \tag{A.3}$$

Of course, our aim is to write the coproducts. Actually, for  $I^+$  we use (4.5) of [14] (since  $I^+$  is expressed through  $B$  which has the (parameter-independent) classical coproduct (3.21b) as in the one-parameter case) which in our notation gives:

$$\delta_{\mathcal{U}}(I_+) = I_+ \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes I_+ + \sum_{n=1}^{\infty} \left(-\frac{\tilde{g}^2}{4}\right)^n (I_+^{n+1} \otimes I_+^n + I_+^n \otimes I_+^{n+1}). \tag{A.4}$$

For the co-product of  $H$  we need the inverse of (A.1a) (cf [13, equation (3.1)]):

$$K^{\pm 1} = e^{\pm \tilde{g}B} = 1_{\mathcal{U}} + 2\sum_{\ell=1}^{\infty} \left(\pm \frac{\tilde{g}}{2} I_+\right)^{\ell} \left(= \frac{1_{\mathcal{U}} \pm \frac{\tilde{g}}{2} I_+}{1_{\mathcal{U}} \mp \frac{\tilde{g}}{2} I_+}\right). \tag{A.5}$$

Then we have using (3.21d):

$$\begin{aligned} \delta_{\mathcal{U}}(H) = & H \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes H + 2\sum_{n=1}^{\infty} \left(H \otimes \left(-\frac{\tilde{g}}{2} I_+\right)^n + \left(\frac{\tilde{g}}{2} I_+\right)^n \otimes H\right) \\ & - 2\tilde{h}I_+ \sum_{k=0}^{\infty} \left(\frac{\tilde{g}}{2} I_+\right)^{2k} \otimes A \left(1_{\mathcal{U}} + 2\sum_{\ell=1}^{\infty} \left(-\frac{\tilde{g}}{2} I_+\right)^{\ell}\right). \end{aligned} \tag{A.6}$$

For the coproduct of  $I_-$  we use (3.21c) and:

$$\delta_{\mathcal{U}}(I_-) = \delta_{\mathcal{U}}\left(\cosh\left(\frac{\tilde{g}B}{2}\right)\right) \delta_{\mathcal{U}}(Y) \delta_{\mathcal{U}}\left(\cosh\left(\frac{\tilde{g}B}{2}\right)\right) \tag{A.7a}$$

$$\delta_{\mathcal{U}}\left(\cosh\left(\frac{\tilde{g}B}{2}\right)\right) = \cosh\left(\frac{\tilde{g}B}{2}\right) \otimes \cosh\left(\frac{\tilde{g}B}{2}\right) + \sinh\left(\frac{\tilde{g}B}{2}\right) \otimes \sinh\left(\frac{\tilde{g}B}{2}\right) \tag{A.7b}$$

to obtain:

$$\begin{aligned}
\delta_U(I_-) = & I_- \otimes \sum_{\ell=0}^{\infty} (\ell+1) \left(-\frac{\tilde{g}}{2} I_+\right)^\ell + \sum_{\ell=0}^{\infty} (\ell+1) \left(\frac{\tilde{g}}{2} I_+\right)^\ell \otimes I_- - \frac{\tilde{g}}{2} (I_+ I_- + I_+ I_-) \\
& \otimes \sum_{\ell=1}^{\infty} \ell \left(-\frac{\tilde{g}}{2} I_+\right)^\ell + \frac{\tilde{g}}{2} \sum_{\ell=1}^{\infty} \ell \left(\frac{\tilde{g}}{2} I_+\right)^\ell \otimes (I_+ I_- + I_+ I_-) + \frac{\tilde{g}^2}{4} I_+ I_- I_+ \\
& \otimes \sum_{\ell=2}^{\infty} (\ell-1) \left(-\frac{\tilde{g}}{2} I_+\right)^\ell + \frac{\tilde{g}^2}{4} \sum_{\ell=2}^{\infty} (\ell-1) \left(\frac{\tilde{g}}{2} I_+\right)^\ell \otimes I_+ I_- I_+ \\
& - \tilde{h}^2 \{I_+ \otimes A^2\} \left\{ \sum_{k=0}^{\infty} (k+1) \left(\frac{\tilde{g}}{2} I_+\right)^{2k} \otimes 1_U + \sum_{k=0}^{\infty} \left(\frac{\tilde{g}}{2} I_+\right)^{2k} \otimes \sum_{\ell=1}^{\infty} \left(-\frac{\tilde{g}}{2} I_+\right)^\ell \right. \\
& + \sum_{k=0}^{\infty} (k+1) \left(-\frac{\tilde{g}}{2} I_+\right)^k \otimes \sum_{\ell=1}^{\infty} \ell \left(-\frac{\tilde{g}}{2} I_+\right)^\ell \left. \right\} + \tilde{h} \{1_U \otimes A\} \{[H \otimes 1_U] \\
& \times \left[ \sum_{k=0}^{\infty} \left(\frac{\tilde{g}}{2} I_+\right)^{2k} \otimes 1_U + 1_U \otimes \sum_{\ell=1}^{\infty} (\ell+1) \left(-\frac{\tilde{g}}{2} I_+\right)^\ell \right. \\
& + 2 \sum_{k=1}^{\infty} \left(-\frac{\tilde{g}}{2} I_+\right)^k \otimes \sum_{\ell=1}^{\infty} \ell \left(-\frac{\tilde{g}}{2} I_+\right)^\ell \left. \right] \\
& - 2 \left[ \sum_{k=1}^{\infty} k \left(-\frac{\tilde{g}}{2} I_+\right)^{2k} \otimes 1_U + \sum_{k=1}^{\infty} k \left(-\frac{\tilde{g}}{2} I_+\right)^k \otimes \sum_{\ell=1}^{\infty} \ell \left(-\frac{\tilde{g}}{2} I_+\right)^\ell \right] \left. \right\}. \quad (\text{A.8})
\end{aligned}$$

In the special case  $\tilde{h} = 0$  the coproducts of  $H$  and  $I_-$  coincide with the one-parameter formulae of [13], cf (3.2) and (5.3), resp., (with  $\tilde{g} \mapsto -h$ ). In the special case  $\tilde{g} = 0$  the nonlinear map becomes an identity and naturally the coproducts of  $I_+$ ,  $I_-$ ,  $H$ , coincide with those of  $B$ ,  $Y$ ,  $H$ , resp., cf (4.5b), (4.5c), (4.5d).

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